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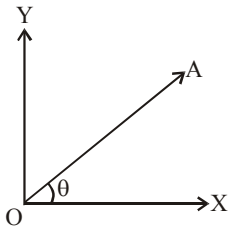
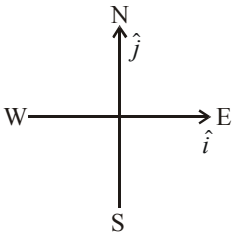
[Vectors]

Important tips & Concepts :

1. **Unit Vector: -** The vector of magnitude unity, used only for direction. $\hat{a} = \frac{\hat{a}}{|\bar{a}|}, \hat{i}, \hat{j}, \hat{k}$ are three mutually $\perp$ unit vectors along X, Y & Z axis resp. The unit vector along $\overline{OA}$ in XY plane is $\hat{OA} = \hat{i}\cos\theta + \hat{j}\sin\theta$ if $\hat{i}, \hat{j}$ denotes East & North resp. then unit vectors –





along N-E is $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$

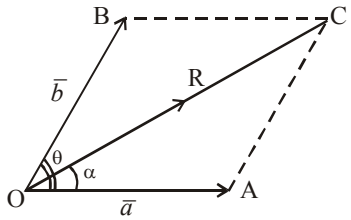
along N-W is $\frac{-\hat{i}+\hat{j}}{\sqrt{2}}$

along S-W is $\frac{-\hat{i}-\hat{j}}{\sqrt{2}}$ etc.

2. **Equal Vectors: -** $\boldsymbol{b} = \boldsymbol{a}$, if (i) $|\boldsymbol{b}| = |\boldsymbol{a}|$, (ii) both are | | along same direction.

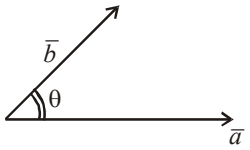
3. **Opposite Vectors: -** $\boldsymbol{b} = -\boldsymbol{a}$, if (i) $|\boldsymbol{b}| = |\boldsymbol{a}|$, (ii) both are | |' but direction are opposite.

4. **Law of Parallelogram of Vectors: -** The resultant of two vectors $\boldsymbol{a}$ & $\boldsymbol{b}$ at an angle θ is $R = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ and R is represented by the diagonal of parallelogram.
The angle α of R with $\boldsymbol{a}$
$\tan\alpha = \frac{b\sin\theta}{a + b\cos\theta}$



$R_{max} = a + b$ when θ = 0
$R_{min} = a - b$ when θ = 180
The resultant of two vectors $\boldsymbol{a}$ & $\boldsymbol{b}$ can lie only between R_{max} & R_{min}

5. **Dot Product of Vectors: -** $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \cos\theta$
$\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b}$
$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1$, if the two vectors are $\perp$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$



6. **Cross Product: -** $\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta\ \hat{n}$
$(\boldsymbol{b} \times \boldsymbol{a}) = -(\boldsymbol{a} \times \boldsymbol{b})$
$\hat{i} \times \hat{j} = \hat{k}, \quad \hat{j} \times \hat{k} = \hat{i}, \quad \hat{k} \times \hat{i} = j$
$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$ if the two vectors are parallel then $\boldsymbol{a} \times \boldsymbol{b} = 0$

7. If $\boldsymbol{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\boldsymbol{b} = b_1\hat{i} + b_2\hat{j} + \hat{b}_3\hat{k}$ then $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$ and

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$= \hat{i}\,(a_2 b_3 - b_2 a_3) - \hat{j}\,(a_1 b_3 - b_1 a_3) + \hat{k}\,(a_1 b_2 - b_1 a_2)$

and if $\boldsymbol{a}$ & $\boldsymbol{b}$ are $\perp^r$ then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$

i.e. $a_1 b_1 + a_2 b_2 + a_3 b_3 = 0$

and if $\boldsymbol{a}$ & $\boldsymbol{b}$ are parallel, then $\boldsymbol{a} \times \boldsymbol{b} = 0$

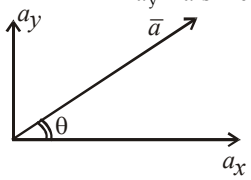
i.e. $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2} = \dfrac{a_3}{b_3}$

8. The unit vector $\perp$ to both $\boldsymbol{a}$ & $\boldsymbol{b}$ is $\hat{n} = \dfrac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|}$

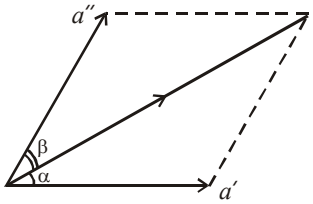
9. Resolution of vector $\boldsymbol{a}$ <u>into $\perp$ Components</u>

$\boldsymbol{a} = a_x \hat{i} + a_y \hat{j}$ where $a_x = a \cos\theta$

$a_y = a \sin\theta$



10. Resolution of vector in any direction



$$\frac{a'}{\sin\beta} = \frac{a''}{\sin\alpha} = \frac{a}{\sin(180 - \alpha + \beta)}$$

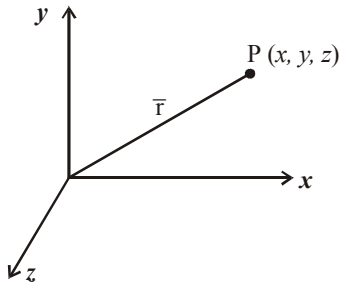
i.e. $a' = \dfrac{a \sin\beta}{\sin(\alpha + \beta)}$, $a'' = \dfrac{a \sin\alpha}{\sin(\alpha + \beta)}$

11. Position Vector of a point P (x, y, z) in space

$\boldsymbol{OP} = \boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\boldsymbol{OP}| = \sqrt{x^2 + y^2 + z^2} = r$

$O\hat{P} = \dfrac{\overline{OP}}{|\overline{OP}|}$



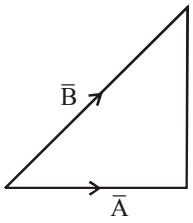
i.e. $O\hat{P} = \dfrac{x}{r}\hat{i} + \dfrac{y}{r}\hat{j} + \dfrac{z}{r}\hat{k}$, direction Cosines of $\boldsymbol{OP} = \boldsymbol{r}$ are $\cos\alpha = \dfrac{x}{r}$, $\cos\beta = \dfrac{y}{r}$,

$\cos\gamma = \dfrac{z}{r}$

and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

12. $\boldsymbol{PQ} = P.V.$ of $Q - P.V$ of P i.e. $\boldsymbol{PQ} = \boldsymbol{OQ} - \boldsymbol{OP}$

13. Applications of Product in Vectors: -



(a) Work = $\boldsymbol{F} \cdot \boldsymbol{d}$

(b) Torque $\tau = \boldsymbol{r} \times \boldsymbol{F}$

(c) Area of $\Delta = \dfrac{1}{2}\,|\boldsymbol{A} \times \boldsymbol{B}|$

(d) Area of parallelogram = $|\boldsymbol{A} \times \boldsymbol{B}|$ where $\boldsymbol{A}$ & $\boldsymbol{B}$ are Consecutive sides of parallelogram

(e) Area of parallelogram = $\dfrac{1}{2}\,|\boldsymbol{C} \times \boldsymbol{D}|$ of $\boldsymbol{C}$ & $\boldsymbol{D}$ are Diagonals of parallelogram

Conceptual Questions (Screening Special)

1. A vector has both magnitude and direction. Does that mean – anything that has magnitude and direction is necessarily a vector.

2. For any arbitrary motion in space, which of the following relations are true:
   (a) $\vec{v}_{av} = \frac{\vec{v}(t_1)+\vec{v}(t_2)}{2}$ (b) $\vec{v}_{av} = \frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1}$
   (c) $\vec{v}(t) = \vec{v}(0) + a\vec{t}$
   (d) $\vec{r}(t) = \vec{r}(0) + \vec{v}(0)t + \frac{1}{2}a\vec{t}^{\,2}$
   (e) $\vec{a}_{av} = \frac{\vec{v}(t_2)-\vec{v}(t_1)}{t_2-t_1}$

3. Can a force directed east ever equal to force directed south?

4. Can a vector be zero when of its components is not zero while all the other components are zero?

5. Is velocity of light a vector or a scalar?

6. Which of the following quantities are independent of the choice of orientation of the co-ordinate axes:
   $\vec{A}+\vec{B}, 3A_x + 2B_y, \left|\vec{A}+\vec{B}-\vec{C}\right|$, angle between $\vec{B}$ and $\vec{C}$ and $\lambda\vec{A}$. Where λ is a scalar.

7. Prove that $\vec{A}\cdot(\vec{A}\times\vec{B}) = 0$.

8. If $\vec{A}, \vec{B}, \vec{C}$ are mutually perpendicular, show that $\vec{C}\times(\vec{A}\times\vec{B}) = 0$. Is the converse true?

9. A vector is not changed if
   (a) It is rotated through an arbitrary angle
   (b) It is multiplied by an arbitrary scalar
   (c) It is cross multiplied by an unit vector
   (d) It is slid parallel to itself.

11. If three vectors add up to zero, must they all be in the same plane?

PART – A

1. In the figure, two vectors ***A*** and ***B*** making an angle θ with each other are shown. Draw a diagram showing $2\boldsymbol{A} - \boldsymbol{B}$.

2. A vector ***P*** has magnitude 2 and another vector ***Q*** has mag. 3. They are ⊥ each other. Find the magnitude of $2\boldsymbol{P} + \boldsymbol{Q}$ and draw the diagram, find its direction also.

3. It is possible that the magnitude of the sum of two vectors of equal magnitudes is equal to the magnitude of each vector?

4. A body is moving with a uniform velocity 10 m/s on a circular path of 2·0 m diameter. Calculate
   (i) The diff. between the displacement and distance covered in half round
   (ii) The magnitude of change in velocity (a) in half round. (b) in one-fourth round.

5. The resultant vector of ***P*** & ***O*** is ***R***. on reversing the direction of ***Q***, the resultant vector becomes ***S*** show that $R^2 + S^2 = 2(P^2 + Q^2)$

6. The resultant of two vectors ***u*** & ***v*** is perpendicular to vector ***u*** and its magnitude is equal to half of the magnitude of vector ***v***, find out the angle between ***u*** & ***v***.

7. The velocity of a body is 100 km/hr, 30º west of south. Find the North & East Component by drawing vector diagram.

8. The *x* and *y* Components of vector ***A*** are 4 and 6 meters. The *x* and *y* Comp. of vector (***A*** + ***B***) are 10 meters and 9 meters resp. Calculate for vector ***B*** (a) its *x* and *y* Components (b) its length (c) the angle it makes with *x*-axis.

9. Find the unit vector $\perp$ to both $\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}$.

10. If $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$ and magnitude of $\boldsymbol{A}$,$\boldsymbol{B}$ & $\boldsymbol{C}$ are 5, 4 & 3 units. Find the angle between $\boldsymbol{A}$ & $\boldsymbol{C}$

11. A person moves 30m North, then 20m East then $30\sqrt{2}$ m South-West, Find his displacement.

12. If $\boldsymbol{F} = 4\hat{i}-3\hat{j}$. The vector $\perp$ to $\boldsymbol{F}$ is ……
(a) $4\hat{i}+3\hat{j}$ (b) $6\hat{i}$ (c) $7\hat{k}$ (d) $3\hat{i}-4\hat{j}$

13. A particle is moving eastward with a velocity of 5 m/s. In 10 sec the velocity changes to 5 m/s, northwards. Find the average acceleration in this time.

14. The point of application of a force $\boldsymbol{F} = 5\hat{i}-3\hat{j}+2\hat{k}$ is moved from $\boldsymbol{r}_1 = 2\hat{i}+7\hat{j}+2\hat{k}$ to $\boldsymbol{r}_2 = 5\hat{i}+2\hat{j}+3\hat{k}$. The work done is ……

15. A force $\boldsymbol{F} = 4\hat{i}-5\hat{j}+3\hat{k}$ is acting at a point $\boldsymbol{r}_1 = \hat{i}+2\hat{j}+3\hat{k}$. The torque acting about a point $\boldsymbol{r}_2 = 3\hat{i}-2\hat{j}-3\hat{k}$ is ……

16. Three vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ satisfy the relation $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ and $\boldsymbol{A}\cdot\boldsymbol{C} = 0$, the vector $\boldsymbol{A}$ is parallel to ……
(a) $\boldsymbol{B}$ (b) $\boldsymbol{C}$ (c) $\boldsymbol{B}\cdot\boldsymbol{C}$ (d) $\boldsymbol{B}\times\boldsymbol{C}$)

17. If $\boldsymbol{A} = \hat{i}\ A\cos\theta - \hat{j}A\sin\theta$ be any vector. Another vector $\boldsymbol{B}$, which is normal to $\boldsymbol{A}$ can be expressed as
(a) $\hat{i}\,\boldsymbol{B}\cos\theta - \hat{j}\,\boldsymbol{B}\sin\theta$
(b) $\hat{i}\,\boldsymbol{B}\cos\theta + \hat{j}\,\boldsymbol{B}\sin\theta$
(c) $\hat{i}\,\boldsymbol{B}\sin\theta - \hat{j}\,\boldsymbol{B}\cos\theta$
(d) $\hat{i}\,\boldsymbol{B}\sin\theta + \hat{j}\,\boldsymbol{B}\cos\theta$

18. If $\boldsymbol{P} = \hat{i}+2\hat{j}+3\hat{k}$ and $\boldsymbol{Q} = 3\hat{i}-2\hat{j}+\hat{k}$ then find the area of parallelogram of sides $\boldsymbol{P}$ & $\boldsymbol{Q}$.

19. If $\boldsymbol{A} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\boldsymbol{b} = 4\hat{i}+3\hat{j}+2\hat{k}$, find the angle between $\boldsymbol{a}$ & $\boldsymbol{b}$.

20. If $\boldsymbol{A} = 2\hat{i}-3\hat{j}+7\hat{k}$, $\boldsymbol{B} = \hat{i}+2\hat{k}$, $\boldsymbol{C} = \hat{j}-\hat{k}$. Find $\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})$.

21. $\hat{i}$ & $\hat{j}$ are unit vectors along x and y axis resp. What is the magnitude and direction of the vectors $\hat{i}+\hat{j}$ and $\hat{i}-\hat{j}$. What are the Components of vector $\boldsymbol{A} = 2\hat{i}+3\hat{j}$ along the direction of $\hat{i}+\hat{j}$ and $\hat{i}-\hat{j}$?

22. Establish the following vector inequalities -
(a) $|\boldsymbol{a}+\boldsymbol{b}| \le |\boldsymbol{a}| + |\boldsymbol{b}|$ (b) $|\boldsymbol{a}+\boldsymbol{b}| \ge |\boldsymbol{a}| - |\boldsymbol{b}|$
(c) $|\boldsymbol{a}-\boldsymbol{b}| \le |\boldsymbol{a}| - |\boldsymbol{b}|$ (d) $|\boldsymbol{a}-\boldsymbol{b}| \ge |\boldsymbol{a}| - |\boldsymbol{b}|$

23. Three vectors are given by $\boldsymbol{a} = 3\hat{i}+3\hat{j}-2\hat{k}$, $\boldsymbol{b} = -\hat{i}-4\hat{j}+2\hat{k}$, $\boldsymbol{c} = 2\hat{i}+2\hat{j}+\hat{k}$ find
(a) $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ (b) $\boldsymbol{a}\cdot(\boldsymbol{b}+\boldsymbol{c})$
(c) $\boldsymbol{a}\times(\boldsymbol{b}+\boldsymbol{c})$

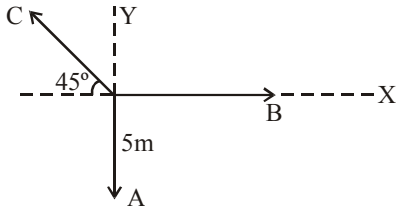
24. At what angle must the two forces ($\boldsymbol{x}+\boldsymbol{y}$) and ($\boldsymbol{x}-\boldsymbol{y}$) act so that the resultant may be $\sqrt{x^2+y^2}$.

25. Surface is -
(a) Scalar
(b) Vector
(c) Neither scalar nor vector
(d) Both scalar and vector

26. Determine the magnitude of vector having initial point P (x_1, y_1, z_1) and terminal point Q(x_2, y_2, z_2).

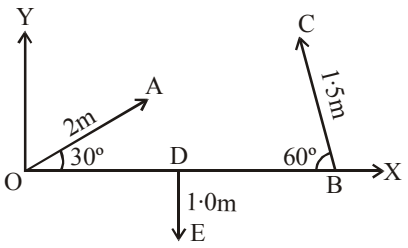
27. The sum of the magnitudes of two forces acting at a point is 18 and the magnitude of their resultant is 12. If resultant is at 90º with the force of smaller magnitude, what are the magnitudes of forces?

28. The sum of the three vectors shown in figure is zero. Find the magnitudes of the vectors $O\vec{B}$ and $O\vec{C}$.



29. Let $\vec{A}$ and $\vec{B}$ be the two vectors of magnitude 10 unit each. If they are inclined to the X-axis at angles 30º and 60º respectively, find the resultant.

30. Refer to figure. Find (a) the magnitude, (b) x and y components and (c) the angle with the X-axis of the resultant of $O\vec{A}$, $B\vec{C}$ and $D\vec{E}$.



31. What are the properties of two vectors **a** and **b** such that
    (a) **a** + **b** = **c** and $a + b = c$;
    (b) **a** + **b** = **a** – **b**
    (c) **a** + **b** = **c** and $a^2 + b^2 = c^2$?

32. A vector **B**, when added to the vector **C** = 3·0 **I** + 4·0 **j**, yields a resultant vector that is in the positive y direction and has a magnitude equal to that of **C**. What is the magnitude of **B**?

33. If $\hat{\alpha}, \hat{\beta}$ are unit vectors and θ is the angle between them show that
    $$\sin\frac{1}{2}\theta = \frac{1}{2}\left|\hat{\alpha} - \hat{\beta}\right|$$

θθθ